November 2, 2026

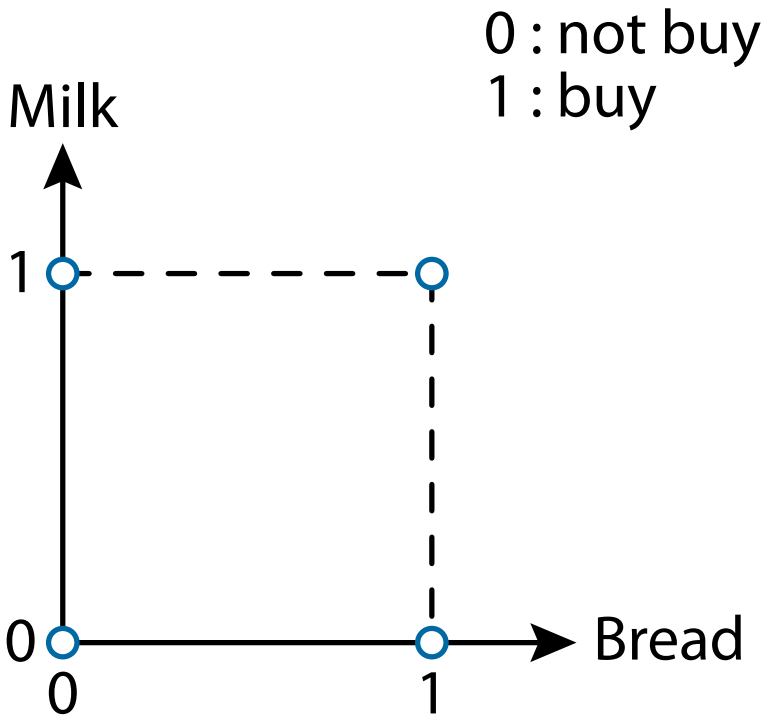
Inter-University Research Institute Corporation /
Research Organization of Information and Systems
National Institute of Informatics

Pattern Discovery and Generative Models

Introduction to Intelligent Systems Science II

Mahito Sugiyama

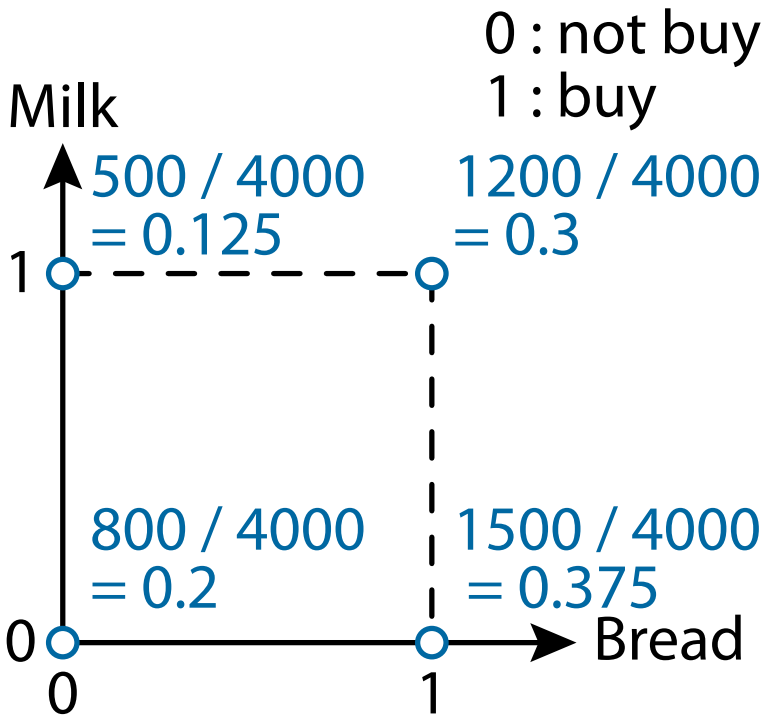
Binary Data → Pattern Mining



- Dataset:

	Bread	Milk
1	0	1
2	1	1
3	1	1
4	1	0
⋮	⋮	⋮
4000	1	0

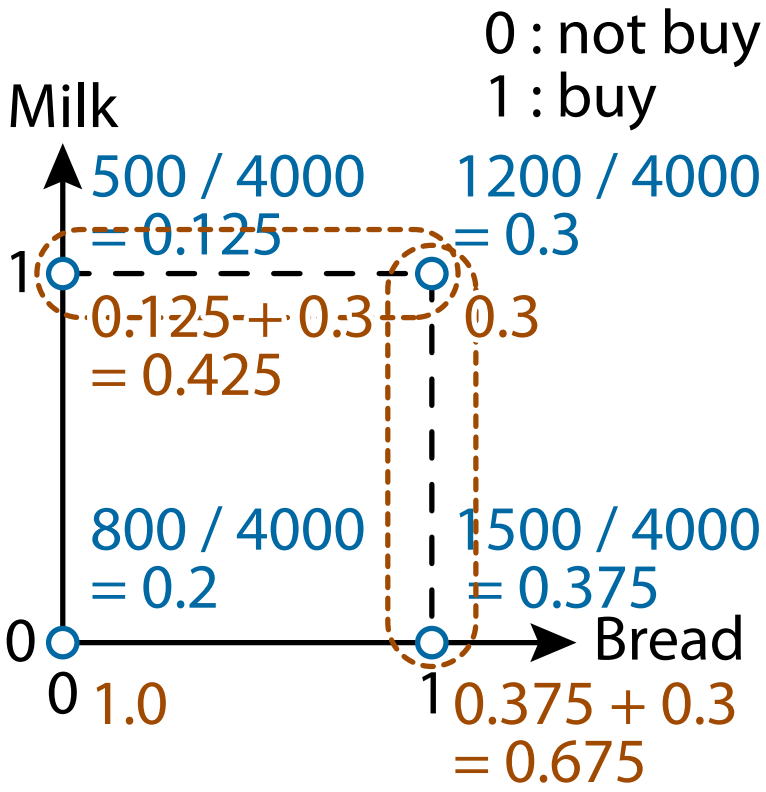
Find Frequent Patterns (Probability)



- Dataset:

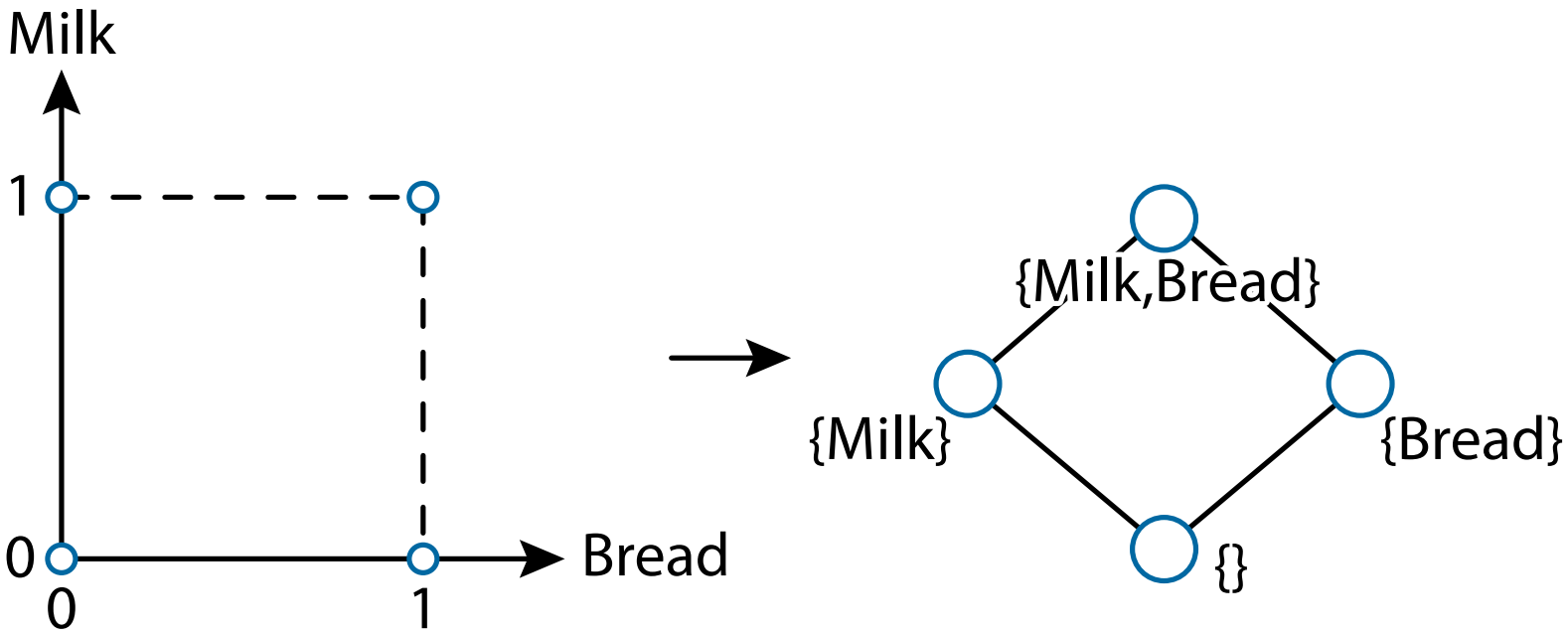
	Bread	Milk
1	0	1
2	1	1
3	1	1
4	1	0
⋮	⋮	⋮
4000	1	0

Find Frequent Patterns (Frequency)

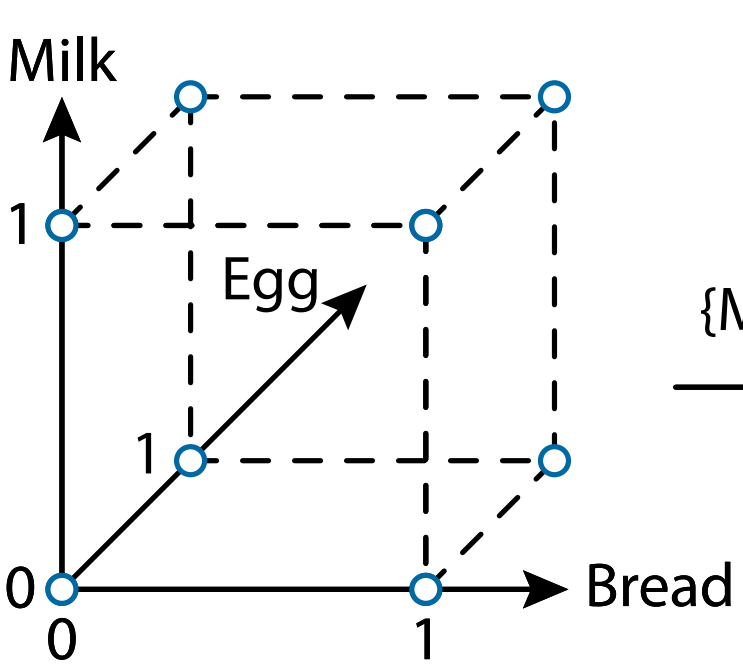


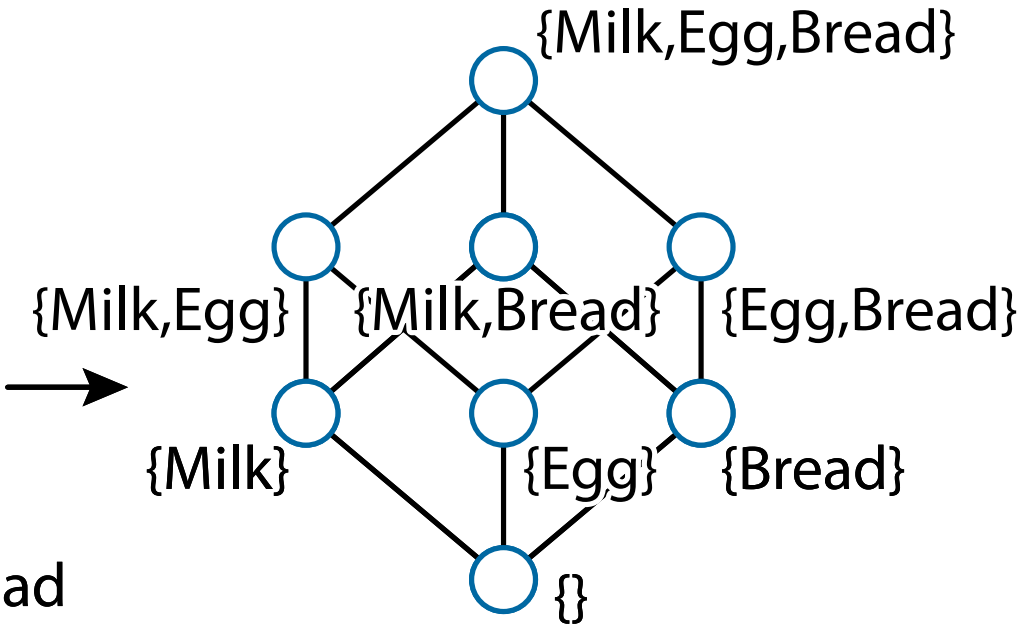
- 67.5% customers bought {bread}
- 42.5% customers bought {milk}
- 30.0% customers boght {bread, milk}
- An item combination is called a pattern

Lattice Representation (2 Variables)

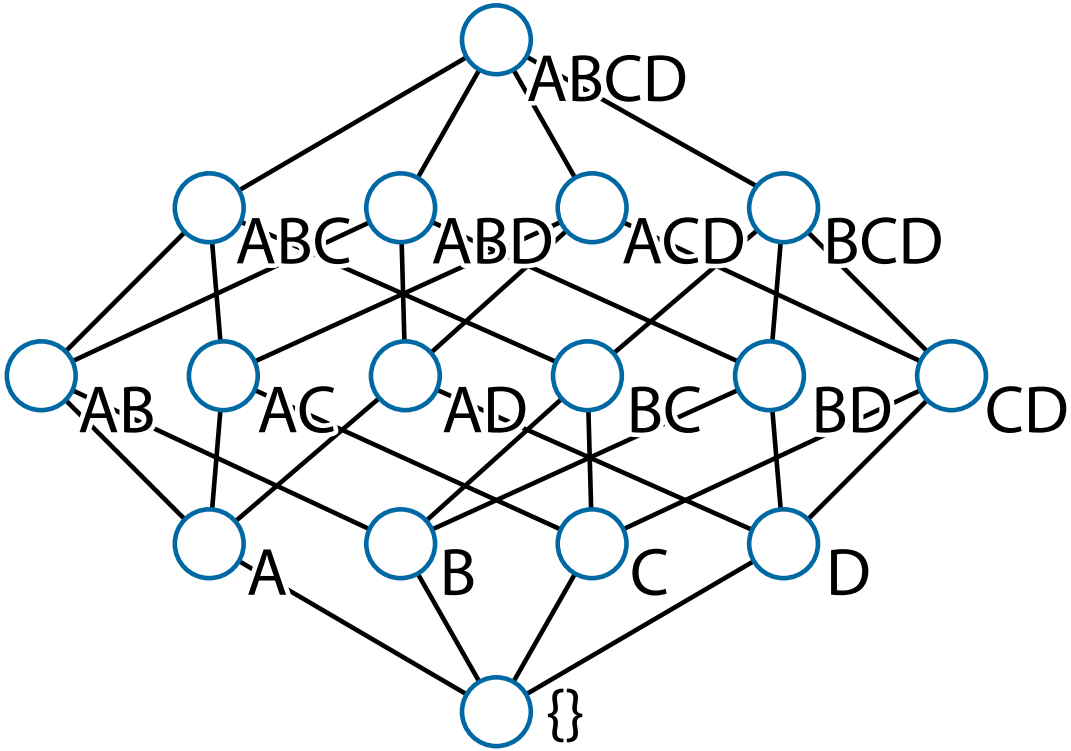


Lattice Representation (3 Variables)





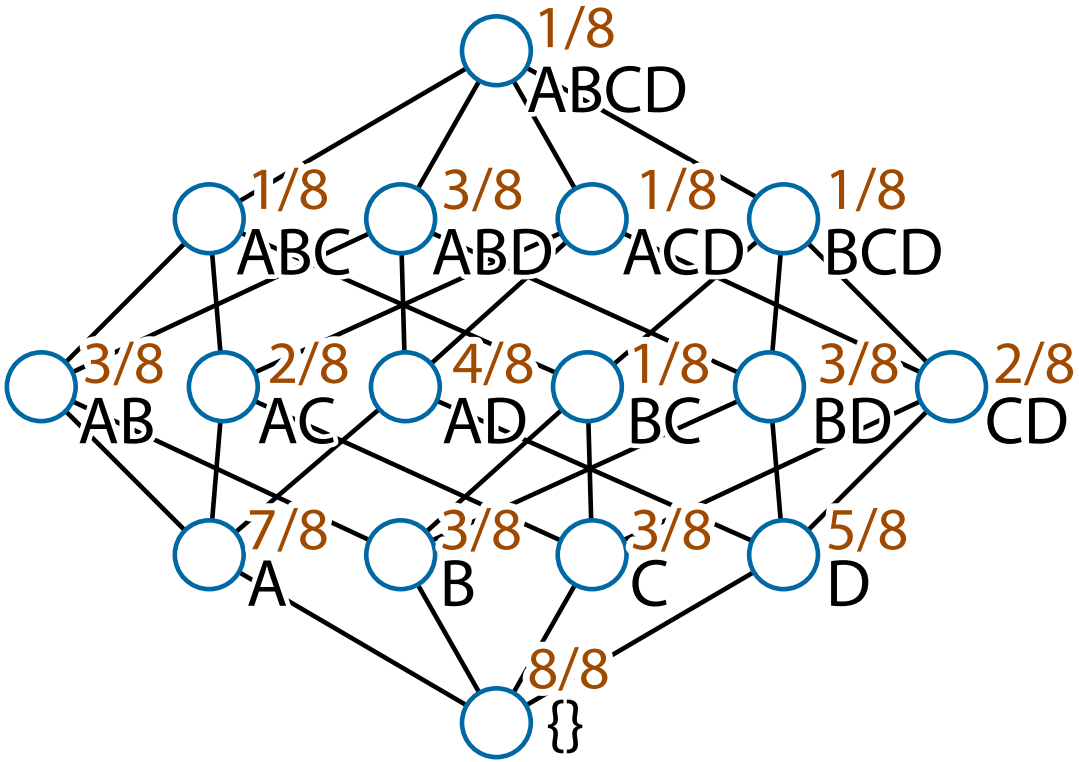
Lattice for 4 Variables



Frequency of Patterns

Dataset:

	A	B	C	D
ID 1:	1	1	0	1
ID 2:	1	0	1	0
ID 3:	1	0	0	0
ID 4:	0	0	1	1
ID 5:	1	1	0	1
ID 6:	1	0	0	0
ID 7:	1	0	0	1
ID 8:	1	1	1	1



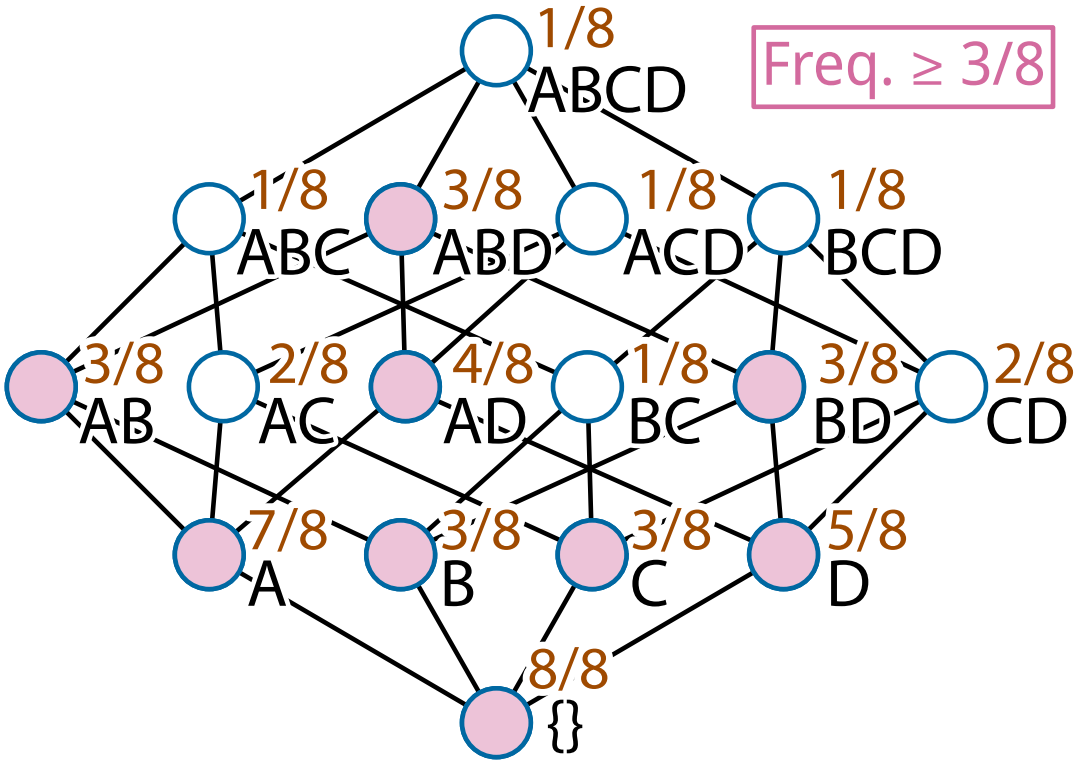
Pattern Mining

- Find frequent patterns (variable combinations)
- Representative methods:
  - Apriori, FP-growth, LCM, ...
- Input: Set of binary vectors $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots \boldsymbol{x}_n\}$, $\boldsymbol{x}_i \in \{0,1\}^d$
  - Multiple occurrence is allowed in X
- Output: Set of patterns (itemsets):
  $\{ s \subseteq \{1, 2, \dots, d\} \mid \eta(s) > \sigma \}$
  - $\eta(s)$ is frequency of s, σ is a threshold given by the user

Search for Frequent Patterns

Dataset:

	A	B	C	D
ID 1:	1	1	0	1
ID 2:	1	0	1	0
ID 3:	1	0	0	0
ID 4:	0	0	1	1
ID 5:	1	1	0	1
ID 6:	1	0	0	0
ID 7:	1	0	0	1
ID 8:	1	1	1	1



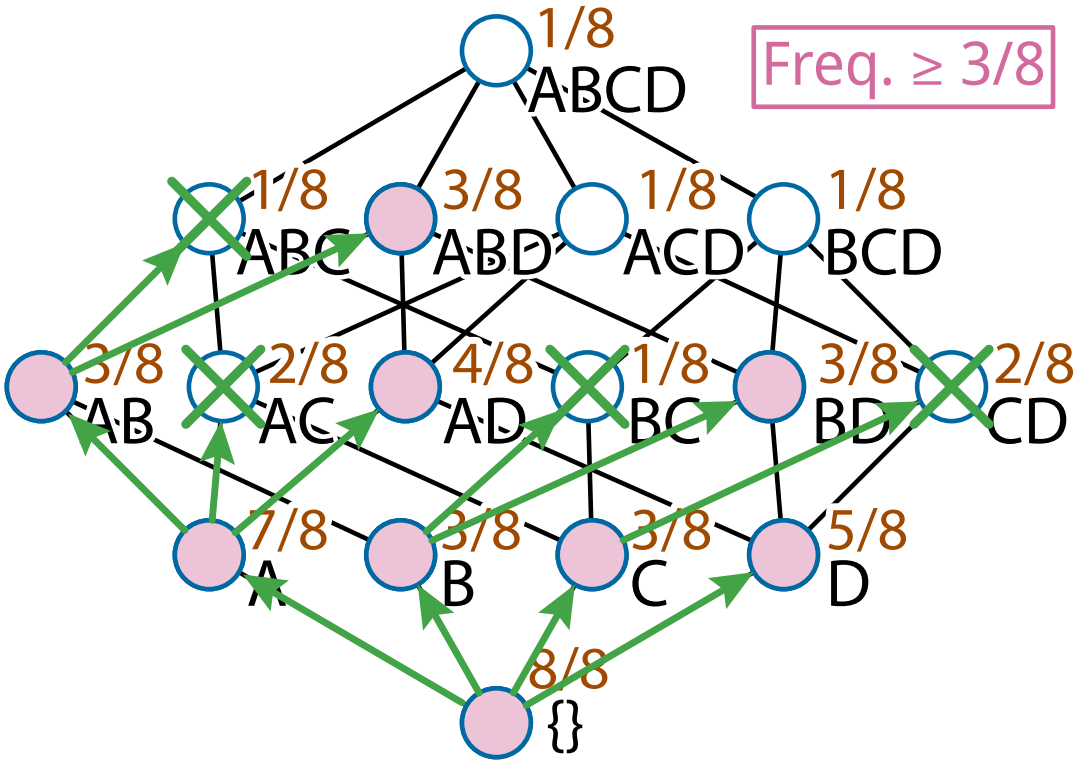
Combinatorial Explosion!! [YouTube]

# var.	# patterns	Runtime in naïve enum.
20	2^{20}	0.00059 sec.
30	2^{30}	0.6 sec.
40	2^{40}	10.2 min.
50	2^{50}	174 hours.
70	2^{70}	7 million days
100	2^{100}	8 thousand billion days

Apriori [Agrawal & Srikant, 1994]

Dataset:

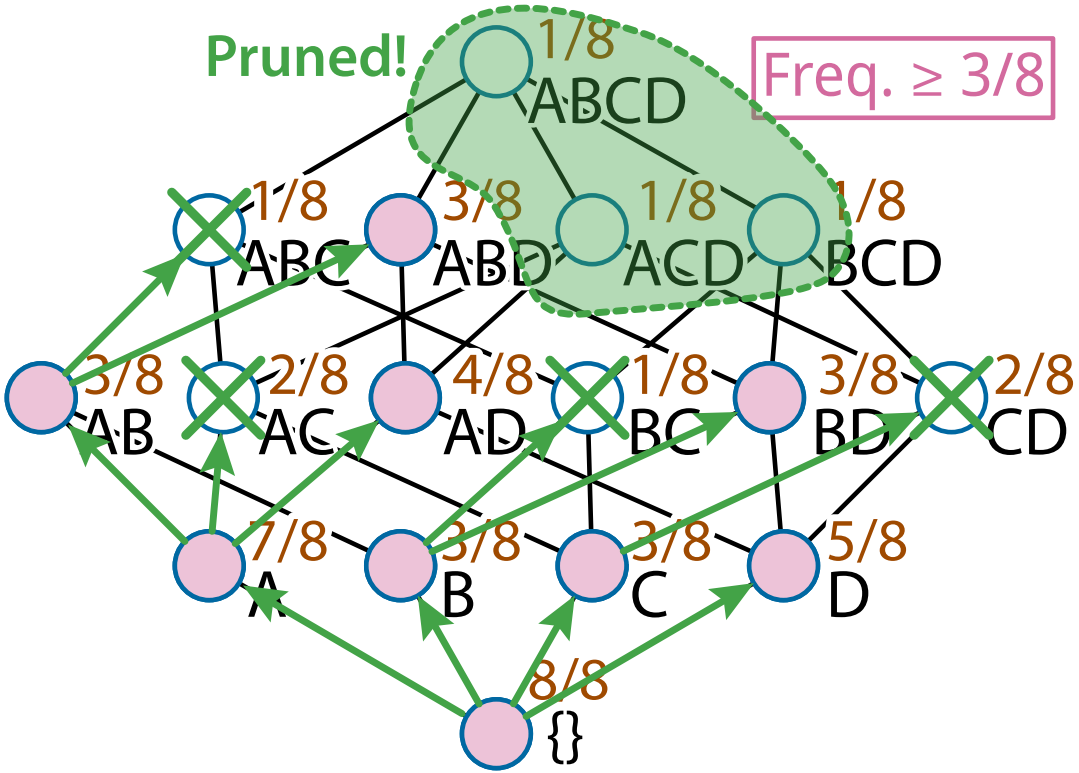
	A	B	C	D
ID 1:	1	1	0	1
ID 2:	1	0	1	0
ID 3:	1	0	0	0
ID 4:	0	0	1	1
ID 5:	1	1	0	1
ID 6:	1	0	0	0
ID 7:	1	0	0	1
ID 8:	1	1	1	1



Apriori [Agrawal & Srikant, 1994]

Dataset:

	A	B	C	D
ID 1:	1	1	0	1
ID 2:	1	0	1	0
ID 3:	1	0	0	0
ID 4:	0	0	1	1
ID 5:	1	1	0	1
ID 6:	1	0	0	0
ID 7:	1	0	0	1
ID 8:	1	1	1	1



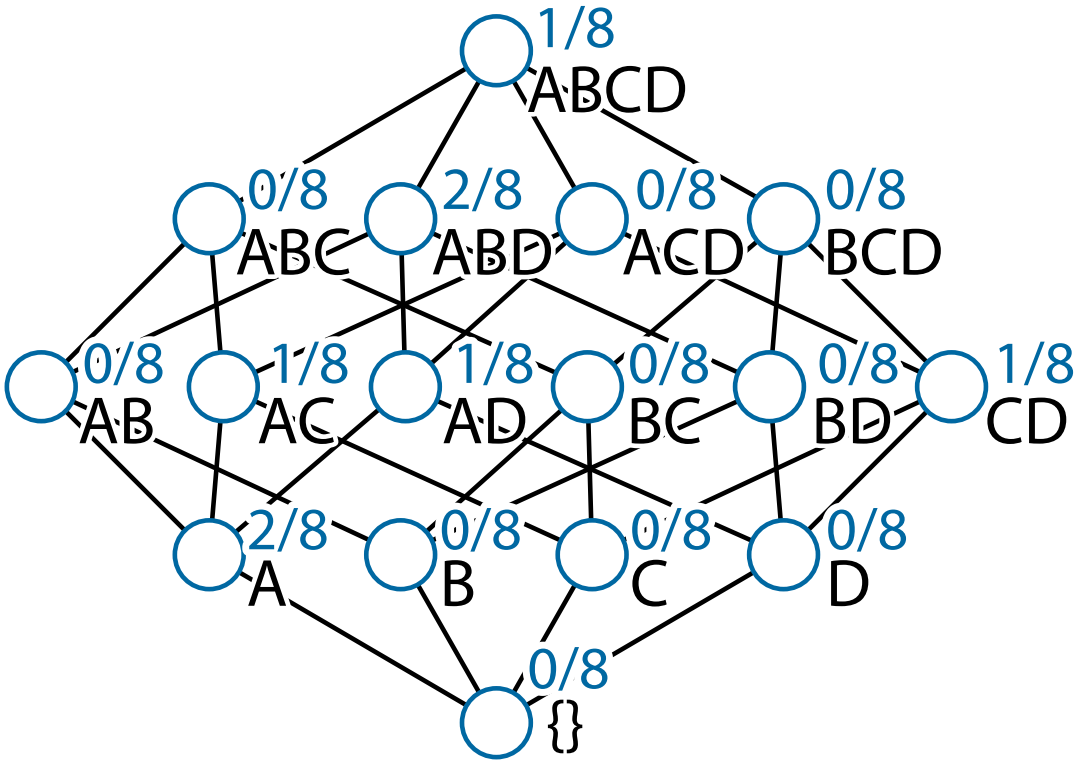
Algorithm 1: Apriori algorithm

```
1 PatternMining(σ)
2     PatternEnumeration (⊥, σ)
3 PatternEnumeration(x, σ)
4     foreach s ⊃ x and |s| = |x| + 1 do
5         if η(s) ≥ σ then
6             Output s
7             PatternEnumeration (s, σ)
```

Probability Distribution on Lattice

Dataset:

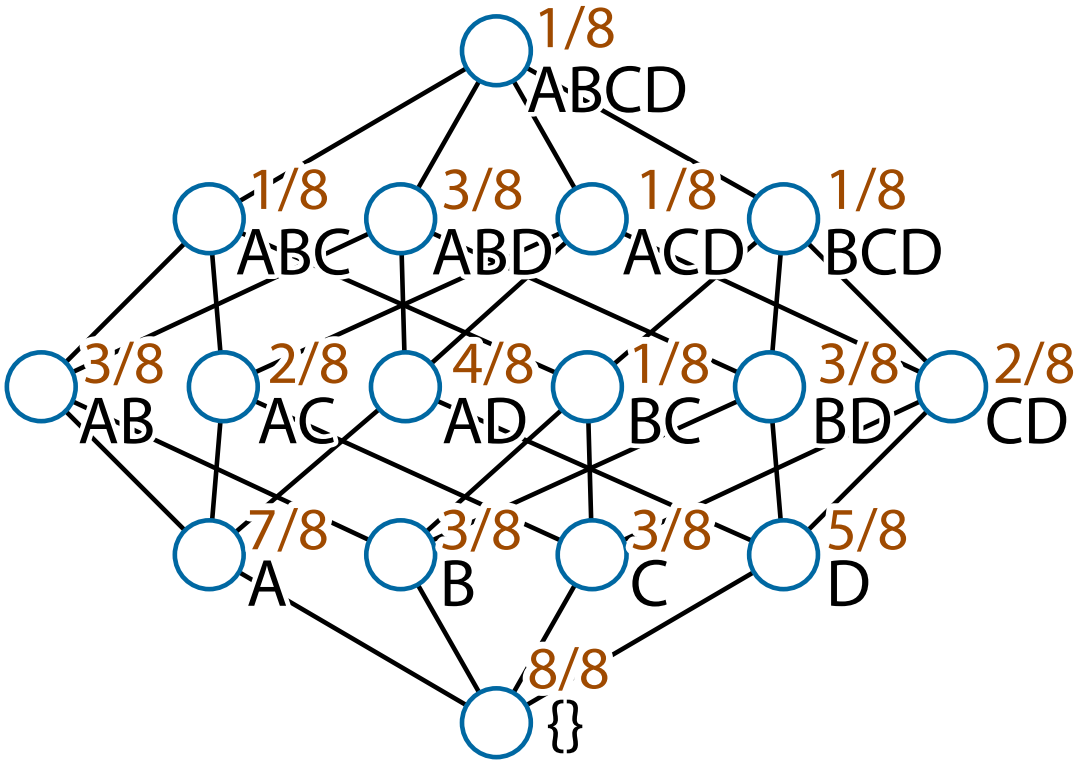
	A	B	C	D
ID 1:	1	1	0	1
ID 2:	1	0	1	0
ID 3:	1	0	0	0
ID 4:	0	0	1	1
ID 5:	1	1	0	1
ID 6:	1	0	0	0
ID 7:	1	0	0	1
ID 8:	1	1	1	1



Probability And Frequency

Dataset:

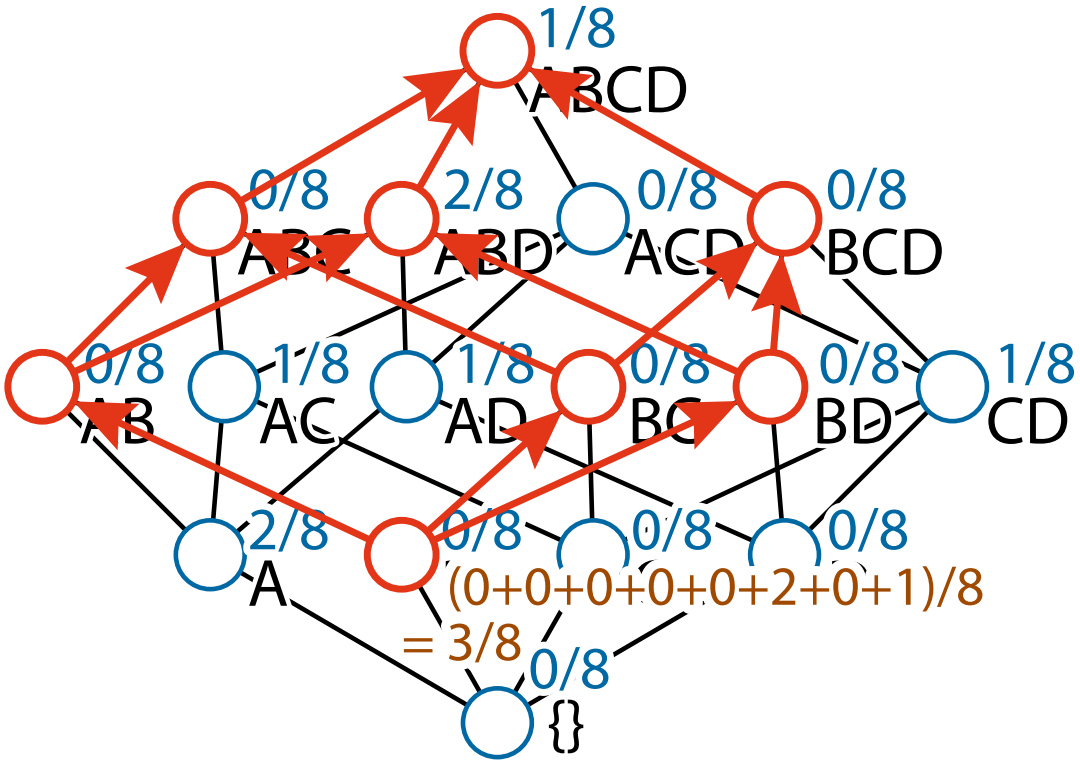
	A	B	C	D
ID 1:	1	1	0	1
ID 2:	1	0	1	0
ID 3:	1	0	0	0
ID 4:	0	0	1	1
ID 5:	1	1	0	1
ID 6:	1	0	0	0
ID 7:	1	0	0	1
ID 8:	1	1	1	1



Sum of prob. in Upper Set = Frequency

Dataset:

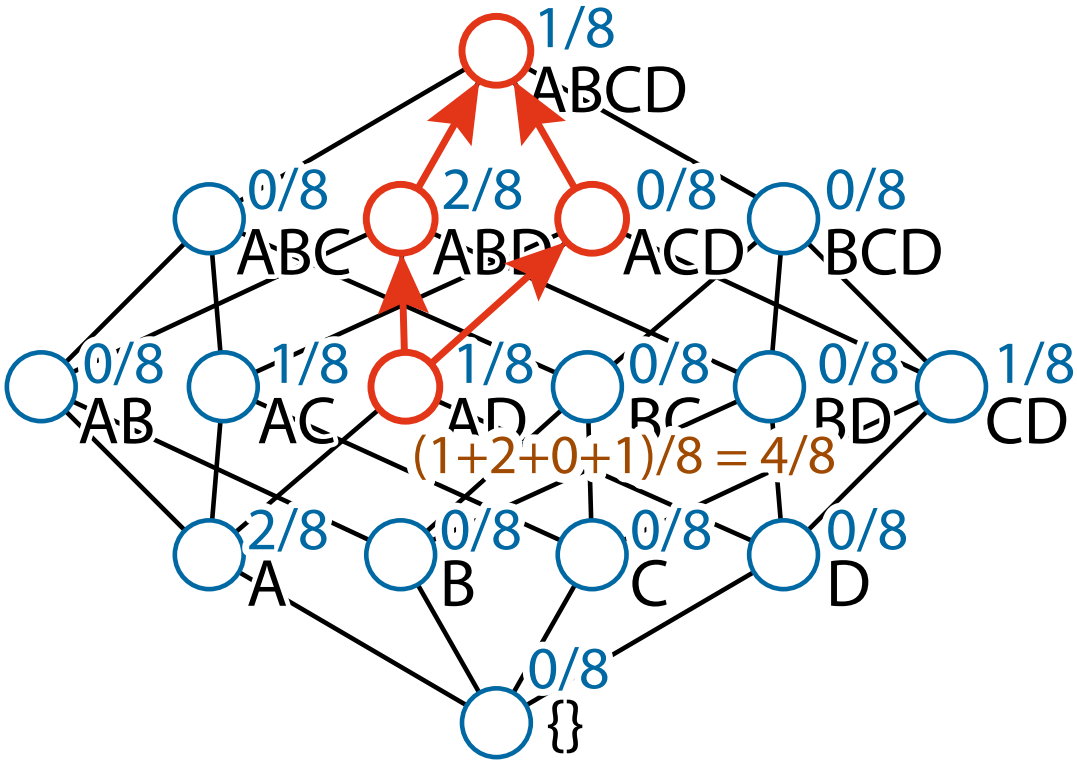
	A	B	C	D
ID 1:	1	1	0	1
ID 2:	1	0	1	0
ID 3:	1	0	0	0
ID 4:	0	0	1	1
ID 5:	1	1	0	1
ID 6:	1	0	0	0
ID 7:	1	0	0	1
ID 8:	1	1	1	1



Sum of prob. in Upper Set = Frequency

Dataset:

	A	B	C	D
ID 1:	1	1	0	1
ID 2:	1	0	1	0
ID 3:	1	0	0	0
ID 4:	0	0	1	1
ID 5:	1	1	0	1
ID 6:	1	0	0	0
ID 7:	1	0	0	1
ID 8:	1	1	1	1



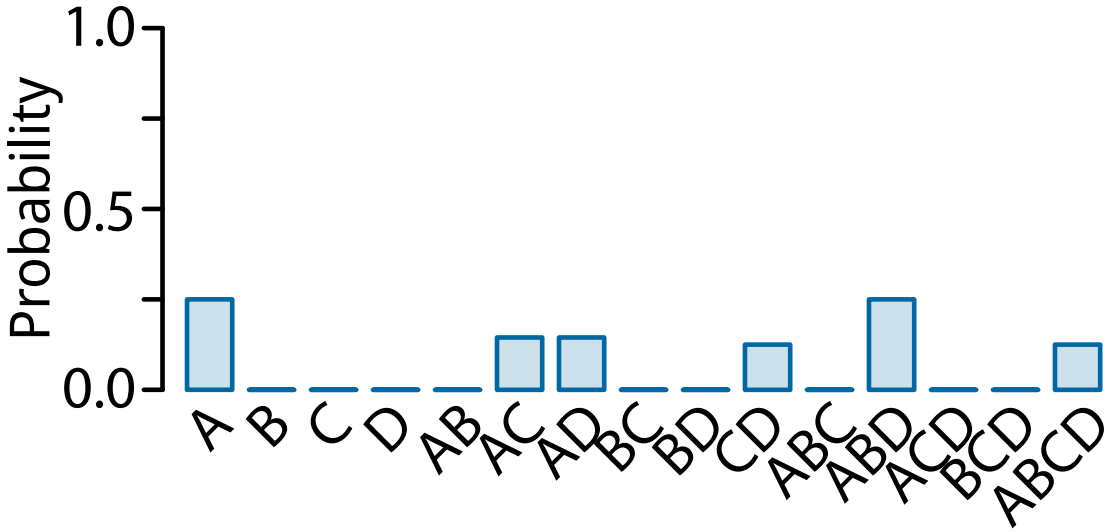
Estimation of Probability Distribution

Dataset:

	A	B	C	D
ID 1:	1	1	0	1
ID 2:	1	0	1	0
ID 3:	1	0	0	0
ID 4:	0	0	1	1
ID 5:	1	1	0	1
ID 6:	1	0	0	0
ID 7:	1	0	0	1
ID 8:	1	1	1	1

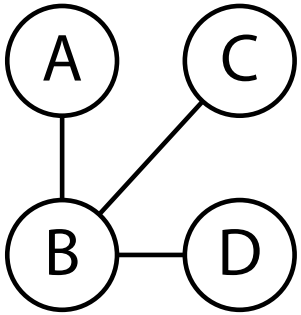
Empirical distribution

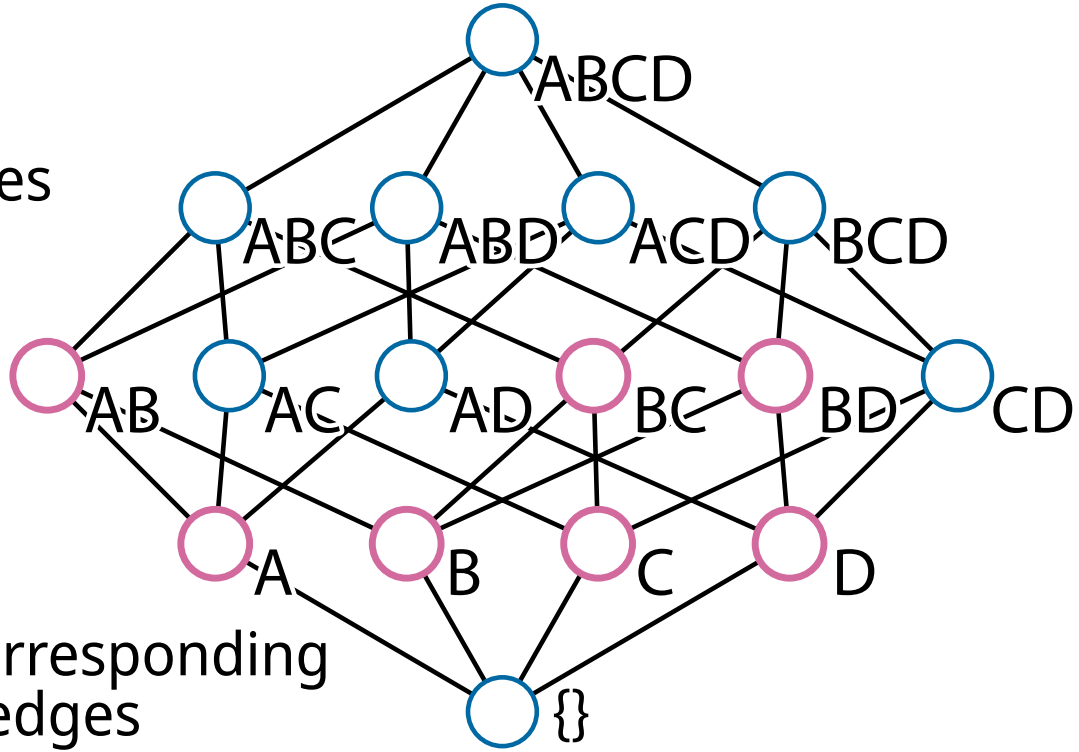
→ What is true distribution?



Boltzmann Machine [Ackley, Hinton & Sejnowski, 1985]

Boltzmann machines

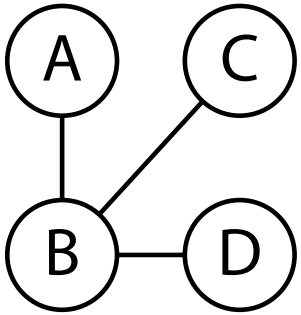


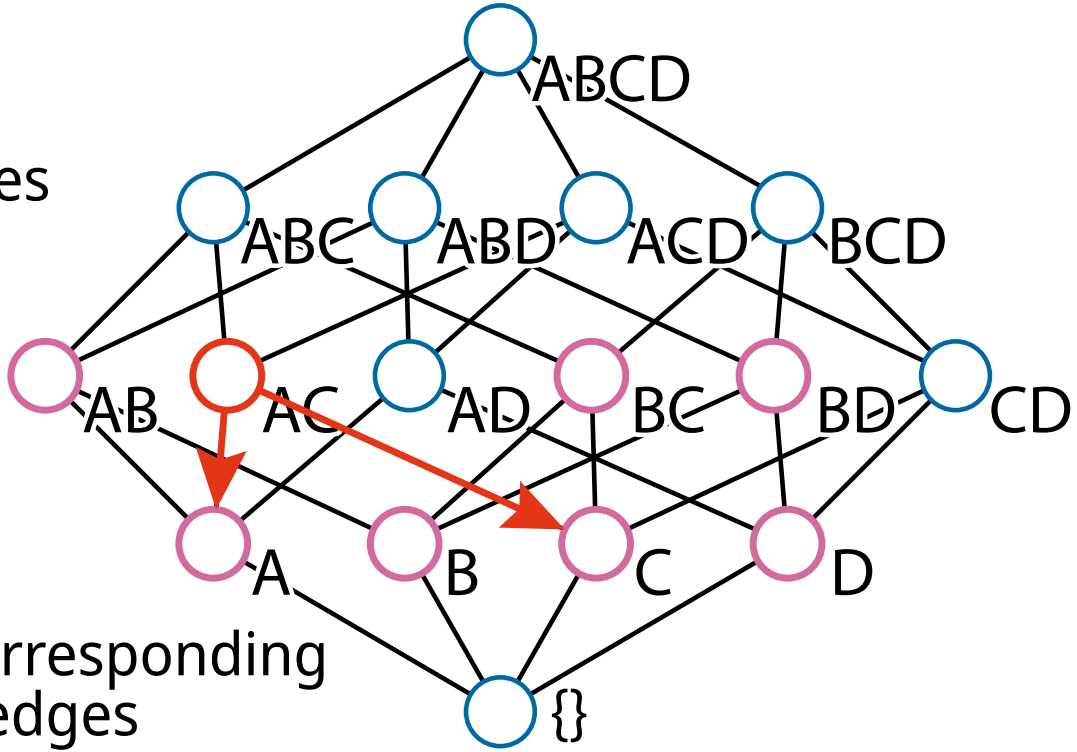


: Parameters corresponding to nodes and edges

Sum of Param. in Lower Set = Probability

Boltzmann machines

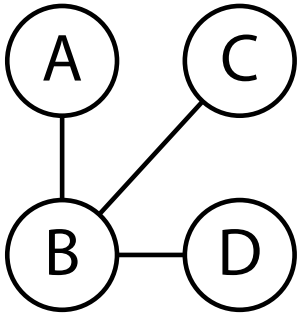


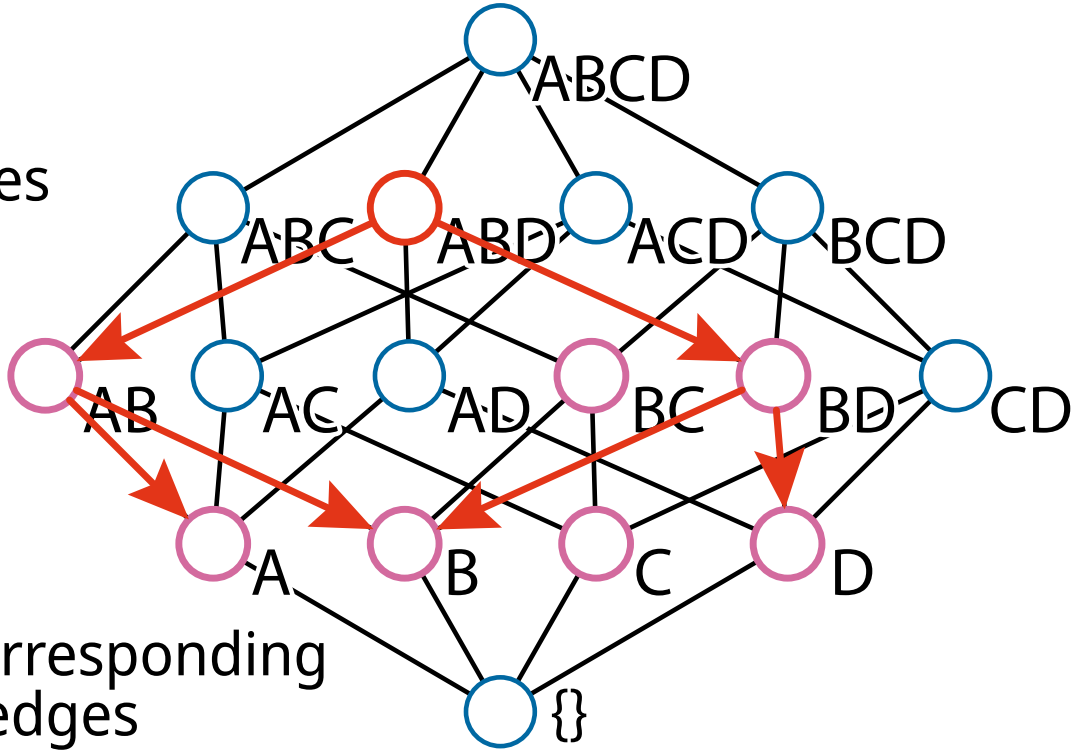


: Parameters corresponding to nodes and edges

Sum of Param. in Lower Set = Probability

Boltzmann machines





: Parameters corresponding to nodes and edges

Probability Computation in BM

- Boltzmann Machine: Undirected graph $G = (V, E)$
  - $V = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$, $E = \{(\mathrm{A}, \mathrm{B}), (\mathrm{B}, \mathrm{C}), (\mathrm{B}, \mathrm{D})\}$
- Parameter: $\boldsymbol{\theta} = (\theta_{\mathrm{A}}, \theta_{\mathrm{B}}, \theta_{\mathrm{C}}, \theta_{\mathrm{D}}, \theta_{\mathrm{AB}}, \theta_{\mathrm{BC}}, \theta_{\mathrm{BD}})$
- Probability of model distribution:

$$p(\mathrm{AC}; \boldsymbol{\theta}) = \exp(\theta_{\mathrm{A}} + \theta_{\mathrm{C}}) / Z$$

$$p(\mathrm{ABD}; \boldsymbol{\theta}) = \exp(\theta_{\mathrm{A}} + \theta_{\mathrm{B}} + \theta_{\mathrm{D}} + \theta_{\mathrm{AB}} + \theta_{\mathrm{BD}}) / Z$$

$$Z = \exp(-\theta_{\emptyset}) \quad \text{(normalizing constant)}$$

Learning of Parameter $\boldsymbol{\theta}$: MLE

- For a dataset $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n\}$,
  find a parameter vector $\boldsymbol{\theta}$ that maximizes likelihood L

  $L_X(\boldsymbol{\theta}) = p(\boldsymbol{x}_1; \boldsymbol{\theta}) \cdot p(\boldsymbol{x}_2; \boldsymbol{\theta}) \cdot \dots \cdot p(\boldsymbol{x}_n; \boldsymbol{\theta})$

- Basic strategy is the gradient method:

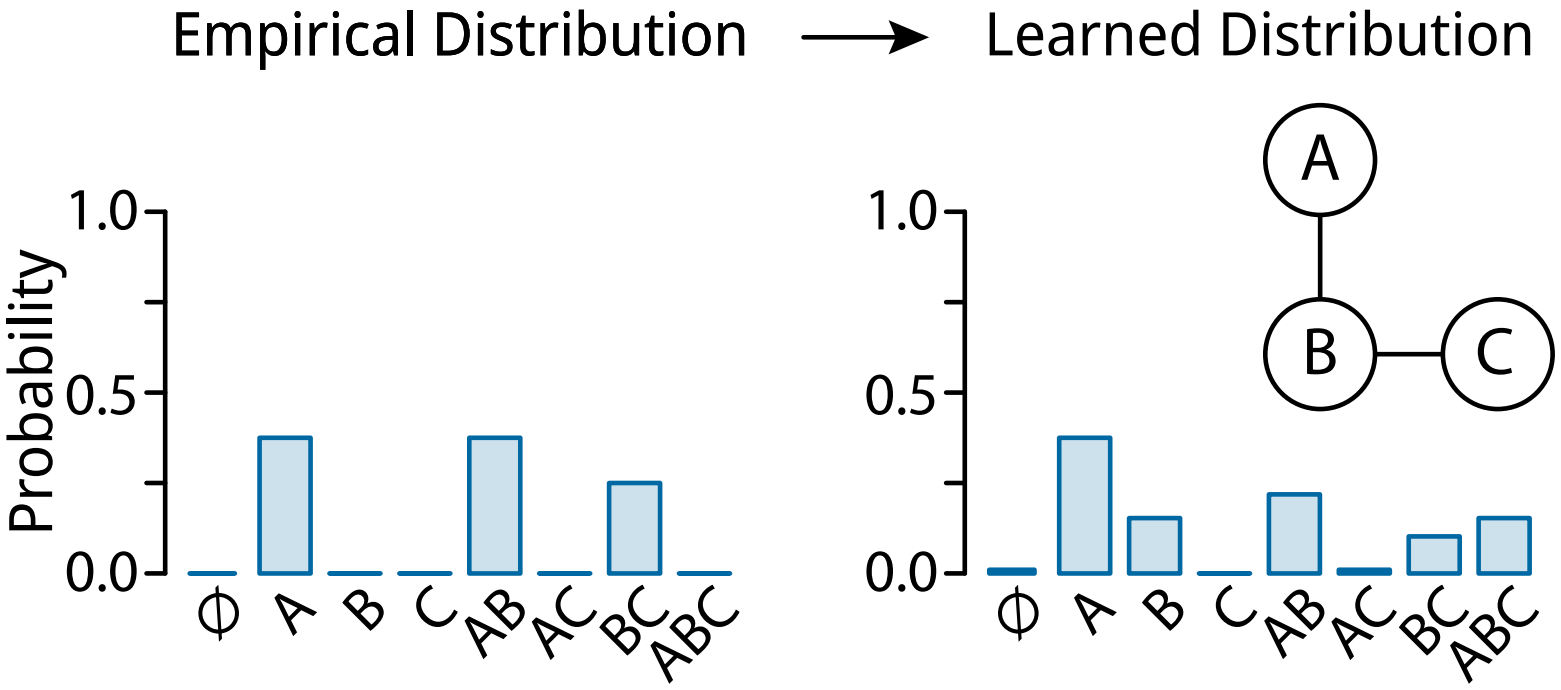
  (i) Start with some $\boldsymbol{\theta}$

  (ii) Compute the direction (gradient) to the goal

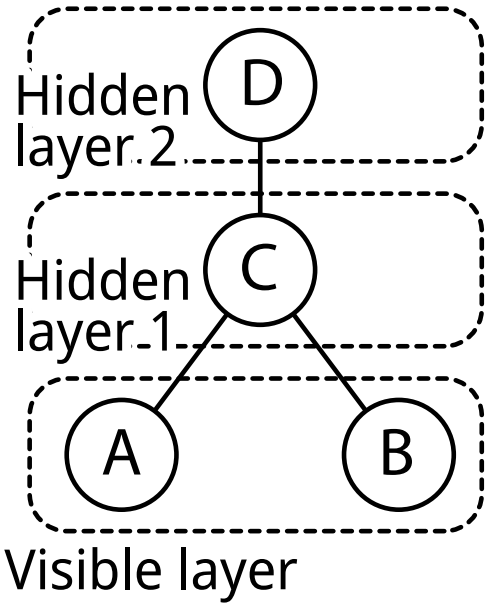
    - Difference between
      frequencies of empirical and those of model distributions

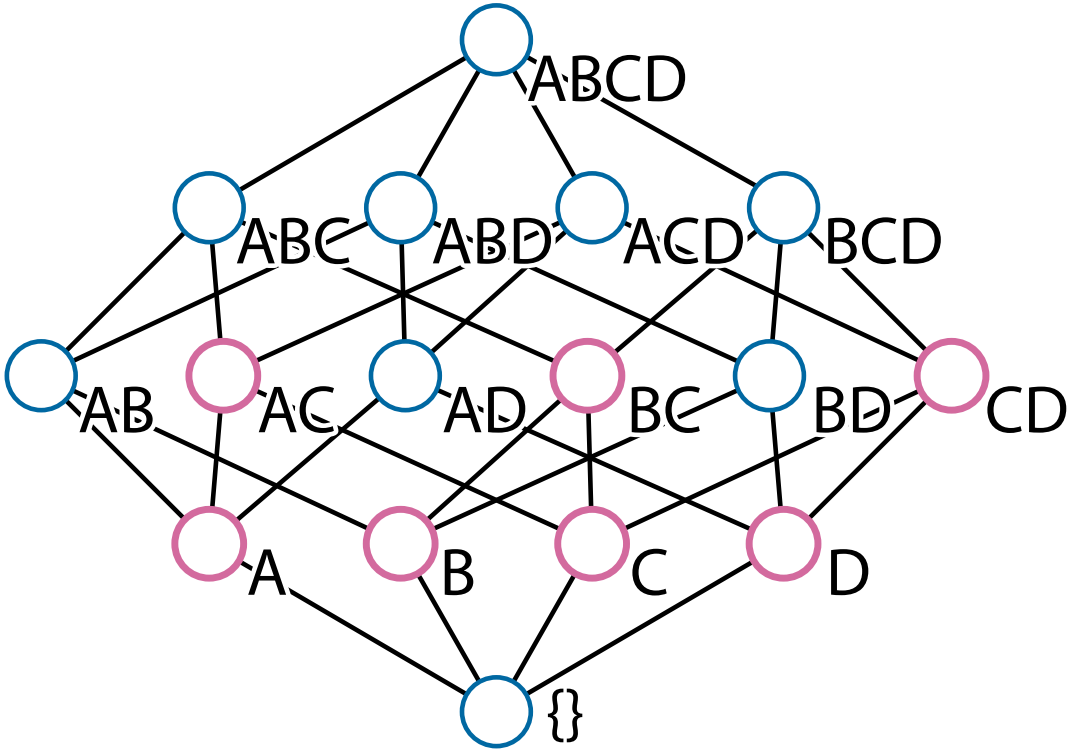
  (iii) Move $\boldsymbol{\theta}$ a bit toward the direction $\rightarrow$ (ii)

Result of BM learning



Deep BM [Salakhutdinov & Hinton, 2009]





Bias-Variance Tradeoff

- Likelihood increases as # parameters increases,
  while a model overfits to data
- Extreme cases:
  - Empirical distribution itself is learned if all parameters are used
    $\rightarrow$ bias = 0, variance is large
  - Uniform distribution is learned if no parameter is used
    $\rightarrow$ bias is large , variance = 0
- There is a tradeoff between bias and variance

Relationship to Info. Geometry

- Consider “the set of probability distributions” (manifold)
- Parameters θ and frequencies η in Boltzmann machines are its coordinate systems
- θ and η are orthogonal, leading to dually flat structure
- The MLE formulation can be applied to various tasks if θ and η are well combined
  - Matrix balancing, tensor decomposition, signal separation, …